SECTION 10.6: ALTERNATING SERIES

EXAMPLE 1: Consider the alternating harmonic series: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$

The divergence test fails (why?) and the integral and comparison tests do not apply (again, why?) Hence, we appeal to the definition of series convergence and examine the sequence of partial sums.

$S_1 = 1$

$S_2 = 1 - \frac{1}{2}$

$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$

$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

$S_6 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

Let's take a closer look at S_6: We may rewrite S_6 as follows: $S_6 = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{5}\right) - \frac{1}{6} < 1.$

Hence S_6 is bounded above by 1. Likewise, we can show $S_n < 1$ for all even indices.

Next, let's write: $S_6 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) = S_2 + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) = S_4 + \left(\frac{1}{5} - \frac{1}{6}\right).$

This shows $S_2 < S_4 < S_6$, a pattern which continues for all even-indexed partial sums.

Hence, the sequence $\{S_2, S_4, S_6, S_8, \ldots\}$ is increasing and bounded above, so it converges to some real number L.

What about the partial sums $\{S_1, S_2, S_3. \ldots\}$? From above, we see that: $S_2 = 1 - \frac{1}{2} = S_1 - \frac{1}{2}$ so that $S_1 = S_2 + \frac{1}{2}$.

Likewise, $S_4 = S_3 - \frac{1}{4}$ so that $S_3 = S_4 + \frac{1}{4}$; $S_6 = S_5 - \frac{1}{6}$ so that $S_5 = S_6 + \frac{1}{6}$, and so on.

In general, $S_{2k-1} = S_{2k} + \frac{1}{2k}$ for $k \geq 1$. Since $\lim_{k\to\infty} S_{2k} = L$ and $\lim_{k\to\infty} \frac{1}{2k} = 0$, we get

$$\lim_{k\to\infty} S_{2k-1} = \lim_{k\to\infty} \left(S_{2k} + \frac{1}{2k}\right) = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k} = L + 0 = L$$

This shows that all the partial sums of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converge to the same number, L, so the series converges.

Distilling what makes the above argument work gives us the following theorem.

ALTERNATING SERIES TEST (AST): Suppose $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is an alternating series (so $a_k > 0$).

If $\{a_k\}$ is a decreasing sequence with $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ **converges**.

NOTE 1: The AST is a test for **convergence**. You can't conclude a series diverges using the AST.

NOTE 2: This can be thought of as a partial converse to the divergence test. (Do you see why?)

EXAMPLE 2: (VIDEO) Determine if the following series converge or diverge:

1. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$

Ans: series converges by the AST.

2. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k}{k^2+1}$

Ans: series converges by the AST.

3. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k^2}{k^2+1}$

Ans: series diverges by the divergence test.

The structure of the series which converge by the AST afford us a (relatively simple!) way to estimate the remainder (error), R_n, incurred by using the partial sum, S_n, to approximate the sum of the series, S.

Since the terms $\{a_n\}$ are decreasing, $a_{n+2} \geq a_{n+3}$, or, written differently: $-a_{n+2} + a_{n+3} \leq 0$.

Hence $a_{n+1} - a_{n+2} + a_{n+3} = a_{n+1} + (-a_{n+2} + a_{n+3}) \leq a_{n+1}$. Continuing in this manner, we get:

$$|S - S_n| = |R_n| = |a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + a_{n+5} + ...| = |a_{n+1} + (-a_{n+2} + a_{n+3}) + (-a_{n+4} + a_{n+5}) + ...| \leq a_{n+1}$$

AST REMAINDER THEOREM: The error incurred by approximating the sum of a series which converges by the AST is bounded by the absolute value of the first neglected term.

EXAMPLE 3: Consider the series: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$.

1. Show the series converges by the AST.

2. Approximate the sum using S_{10}.

Ans: $S_{10} \approx 0.8179$

3. Estimate the error in your approximation in 2. using the AST Remainder Theorem.

Ans: error at most $\dfrac{1}{121} < 0.0083$.

ABSOLUTE AND CONDITIONAL CONVERGENCE

Most of the tests we've seen in this chapter require the series to have positive terms. So it seems natural to study the relationship between a series and the series obtained by taking the absolute value of each term.

DEFINITION: A series $\sum_{k}^{\infty} a_k$ is called **absolutely convergent** if $\sum_{k}^{\infty} |a_k|$ converges.

EXAMPLE 4: Both series below are convergent (why?) Which are absolutely convergent?

1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

Ans: absolutely convergent.

2. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

Ans: not absolutely convergent.

DEFINITION: A series $\sum_{k}^{\infty} a_k$ is called **conditionally convergent** if $\sum_{k}^{\infty} a_k$ converges but $\sum_{k}^{\infty} |a_k|$ diverges.

Hence, the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is conditionally convergent.

One may wonder if it is possible for $\sum_{k}^{\infty} |a_k|$ to converge but $\sum_{k}^{\infty} a_k$ to diverge. The answer is no.

ABSOLUTE CONVERGENCE THEOREM (AC): If $\sum_{k}^{\infty} |a_k|$, then so does $\sum_{k}^{\infty} a_k$.

The proof of AC actually follows from the DCT and properties of convergent sums!

Consider the inequality: $0 \le a_k + |a_k| \le 2\,|a_k|$. If $\sum_{k}^{\infty} |a_k|$ converges, then so does $\sum_{k}^{\infty} 2\,|a_k| = 2\sum_{k}^{\infty} |a_k|$.

By the DCT, $\sum_{k}^{\infty} (a_k + |a_k|)$ converges. Now $a_k = (a_k + |a_k|) - |a_k|$. Since $\sum_{k}^{\infty} (a_k + |a_k|)$ and $\sum_{k}^{\infty} |a_k|$ converge,

$$\sum_{k}^{\infty} a_k = \sum_{k}^{\infty} [(a_k + |a_k|) - |a_k|] = \sum_{k}^{\infty} (a_k + |a_k|) - \sum_{k}^{\infty} |a_k|\,.$$

That is, $\sum_{k}^{\infty} a_k$ converges.

EXAMPLE 5: Determine of the series below are absolutely convergent, conditionally convergent, or divergent.

1. **(VIDEO)** $\displaystyle\sum_{k=0}^{\infty} \frac{(-2)^{2k}}{3^k}$

Ans: divergent

2. **(VIDEO)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k-1}$

Ans: conditionally convergent

3. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(2k)}{k\sqrt{k}}$

Ans: absolutely convergent

Conditionally convergent series possess the following remarkable property:

RIEMANN'S REARRANGEMENT THEOREM: For every real number, L:

If $\displaystyle\sum_{k}^{\infty} a_k$ is conditionally convergent, there is a rearrangement of $\displaystyle\sum_{k}^{\infty} a_k$ which converges to L.

By contrast, **every** rearrangement of an **absolutely convergent** series converges to the **same** number, L.

Paraphrasing, properties like associativity and commutativity of addition are upheld in absolutely convergent series and are completely broken in conditionally convergent ones.

EXAMPLE 6: We'll show in the next chapter that the alternating harmonic series converges to $\ln(2)$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + - \ldots = \ln(2)$$

Note we can rearrange the terms as follows:

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + - \ldots = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + - \ldots$$

Factoring out $\frac{1}{2}$ gives: $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + - \ldots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - \ldots\right) = \frac{1}{2}\ln(2)$

By rearranging the terms of the alternating harmonic sequence, we've arrived at half the original sum!

HOMEWORK: Section 10.6: 11 - 65 odd.